

Propositional Calculus

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Summary

In this text we present the development of propositional calculus in axiomatic form. For background informations see under http://en.wikipedia.org/wiki/Propositional_calculus and <http://www.ltn.lv/~podnieks/mlog/ml1.htm#axioms>.

Chapter 1

Axioms and Rules of Inference

We now state the system of axioms for the propositional calculus and present the rules for obtaining new formulas from them.

1.1 Axioms

Here we just list the axioms without further explanations.

Axiom 1 (Implication Introduction). [axiom:THEN-1]

$$A \rightarrow (B \rightarrow A)$$

Axiom 2 (Distribute Hypothesis over Implication). [axiom:THEN-2]

$$(A \rightarrow (B \rightarrow C)) \rightarrow ((A \rightarrow B) \rightarrow (A \rightarrow C))$$

Axiom 3 (Eliminate Conjunction Right). [axiom:AND-1]

$$(A \wedge B) \rightarrow A$$

Axiom 4 (Eliminate Conjunction Left). [axiom:AND-2]

$$(A \wedge B) \rightarrow B$$

Axiom 5 (Conjunction Introduction). [axiom:AND-3]

$$B \rightarrow (A \rightarrow (A \wedge B))$$

Axiom 6 (Disjunction Introduction Right). [axiom:OR-1]

$$A \rightarrow (A \vee B)$$

Axiom 7 (Disjunction Introduction Left). [axiom:OR-2]

$$A \rightarrow (B \vee A)$$

Axiom 8 (Disjunction Elimination). [axiom:OR-3]

$$(A \rightarrow B) \rightarrow ((C \rightarrow A) \rightarrow ((A \vee C) \rightarrow B))$$

Axiom 9 (Negation Introduction). [axiom:NOT-1]

$$(A \rightarrow B) \rightarrow ((A \rightarrow \neg B) \rightarrow \neg A)$$

Axiom 10 (Negation Elimination). [axiom:NOT-2]

$$\neg A \rightarrow (A \rightarrow B)$$

Axiom 11 (Excluded Middle). [axiom:NOT-3]

$$A \vee \neg A$$

Axiom 12 (Equivalence Elimination right). [axiom:IFF-1]

$$(A \leftrightarrow B) \rightarrow (A \rightarrow B)$$

Axiom 13 (Equivalence Elimination left). [axiom:IFF-2]

$$(A \leftrightarrow B) \rightarrow (B \rightarrow A)$$

Axiom 14 (Equivalence Introduction). [axiom:IFF-3]

$$(A \rightarrow B) \rightarrow ((B \rightarrow A) \rightarrow (A \leftrightarrow B))$$

1.2 First Propositions

Here we draw the first conclusions.

Proposition 1. [proposition:implicationReflexive1]

$$A \rightarrow A$$

Proof.

- | | |
|---|------------------------------------|
| (1) $A \rightarrow (B \rightarrow A)$ | Add axiom 1 |
| (2) $(A \rightarrow (B \rightarrow C)) \rightarrow ((A \rightarrow B) \rightarrow (A \rightarrow C))$ | Add axiom 2 |
| (3) $A \rightarrow (B \vee A)$ | Add axiom 7 |
| (4) $A \rightarrow ((B \vee A) \rightarrow A)$ | SubstPred B by $B \vee A$ in (1) |
| (5) $(A \rightarrow ((B \vee A) \rightarrow C)) \rightarrow ((A \rightarrow (B \vee A)) \rightarrow (A \rightarrow C))$ | SubstPred B by $B \vee A$ in (2) |
| (6) $(A \rightarrow ((B \vee A) \rightarrow A)) \rightarrow ((A \rightarrow (B \vee A)) \rightarrow (A \rightarrow A))$ | SubstPred C by A in (5) |
| (7) $(A \rightarrow (B \vee A)) \rightarrow (A \rightarrow A)$ | MP (6), (4) |
| (8) $A \rightarrow A$ | MP (7), (3) |

□

Proposition 2. [proposition:implication12]

$$((A \rightarrow B) \wedge (B \rightarrow C)) \rightarrow (A \rightarrow C)$$

Proposition 3. [proposition:implication13]

$$(A \rightarrow B) \rightarrow ((B \rightarrow C) \rightarrow (A \rightarrow C))$$

Proposition 4. [proposition:implication14]

$$(A \rightarrow B) \rightarrow ((A \rightarrow C) \rightarrow (A \rightarrow (B \wedge C)))$$

Proposition 5. [proposition:implication15]

$$(A \wedge B) \rightarrow (B \wedge A)$$

Proposition 6. [proposition:implication16]

$$(A \vee B) \rightarrow (B \vee A)$$

Proposition 7. [proposition:implication17]

$$A \rightarrow \neg\neg A$$

Proposition 8. [proposition:implication18]

$$\neg\neg A \rightarrow A$$

Proposition 9. [proposition:implication19]

$$(A \rightarrow (B \rightarrow C)) \rightarrow (B \rightarrow (A \rightarrow C))$$

Proposition 10. [proposition:implication20]

$$(A \rightarrow (B \rightarrow C)) \rightarrow ((A \wedge B) \rightarrow C)$$

Proposition 11. [proposition:implication21]

$$((A \wedge B) \rightarrow C) \rightarrow (A \rightarrow (B \rightarrow C))$$

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