Elements of Mathematical Logic

Michael Meyling

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Summary

The project **Hilbert II** deals with the formal presentation and documentation of mathematical knowledge. For this reason **Hilbert II** provides a program suite to accomplish that tasks. The concrete documentation of mathematical basics is also a purpose of this project. For further informations about the **Hilbert II** project see under http://www.qedeq.org/index_de.html.

This document describes the logical axoims and the rules and meta rules that are used to derive new propositions.

The presentation is axiomatic and in a formal form. A formal calculus is given that enables us to derive all true formulas. Additional derived rules, definitions, abbreviations and syntax extensions basically correspond with the mathematical practice.

This document is also written in a formal language, the original text is a XML file with a syntax defined by the XSD http://www.qedeq.org/current/xml/qedeq.xsd.

This document is work in progress and is updated from time to time. Especially at the locations marked by „+++“ additions or changes will take place.

Foreword

The whole mathematical universium can be unfolded by set–theoretic means. Beside the set–theoretic axioms only logical axioms and rules are required. These elementary basics are sufficient to define the most complex mathematical structures and enable us to prove propositions for those structures. This approach can be fully formalized and can be reduced to simple manipulations of character strings. The semantical interpretation of these character strings represent the mathematical universum.

It is more than convenient to introduce abbreviations and use further derivation rules. But these comforts could be eliminated and replaced by the basic terms at any time[1].

This project has its source in a childhood dream to undertake a formalization of mathematics. In the meantime the technical possibilities are highly developed so that a realisation seems within reach.

Special thanks go to the professors *W. Kerby* and *V. Günther* of the university of Hamburg for their inspiring lectures about logic and axiomatic set theory. Without these important impulses this project would not exist.

I am deeply grateful to my wife *Gesine Dräger* and our son *Lennart* for their support and patience.

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Michael Meyling

[1] At least this is theoretically possible. This transformation is not in each case practically realisable due to restrictions in time and space. For example it is not possible to write down the natural number $1,000,000,000$ completely in set notation.

Introduction

At the beginning we quote *D. Hilbert* from the lecture „The Logical Basis of Mathematics“, September 1922[2].

> „The fundamental idea of my proof theory is the following:
>
> All the propositions that constitute in mathematics are converted into formulas, so that mathematics proper becomes all inventory of formulas. These differ from the ordinary formulas of mathematics only in that, besides the ordinary signs, the logical signs especially „implies“ ($\rightarrow$) and for „not“ ($^-$) occur in them. Certain formulas, which serve as building blocks for the formal edifice of mathematics, are called axioms. A proof is an array that must be given as such to our perceptual intuition of it of inferences according to the schema
>
> $$\frac{A \qquad A \rightarrow B}{B}$$
>
> where each of the premises, that is, the formulae, A and $A \rightarrow B$ in the array either is an axiom or directly from an axiom by substitution, or else coincides with the end formula B of an inference occurring earlier in the proof or results from it by substitution. A formula is said to be provable if it is either an axiom or the end formula of a proof.“

In the 1928 published book *Grundzüge der theoretischen Logik* (Principles of Theoretical Logic) *D. Hilbert* and *W. Ackermann* formalized propositional calculus in a way that build the basis for the logical system used here. 1959 *P. S. Novikov* specified a refined axiom and rule system for predicate calculus.

In this text we present a first order predicate calculus with identity and functors that is the starting basis for the development of the mathematical theory. Only the results without any proofs and in short form are given in the following.

[2]Lecture given at the Deutsche Naturforscher-Gesellschaft, September 1922.

Chapter 1

Language

At the beginning there is logic. Logic is the analysis of methods of reasoning. It helps to derive new propositions from already given ones. Logic is universally applicable.

The logical foundation of **Hilbert II** will be introduced here. The language of our calculus bases on the formalisations of *D. Hilbert*, *W. Ackermann*, *P. Bernays* and *P. S. Novikov*. New rules can be derived from the herein presented. Only these meta rules lead to a smooth flowing logical argumentation.

1.1 Terms and Formulas

We use the *logical symbols* $L = \{$ '$\neg$', '$\vee$', '$\wedge$', '$\leftrightarrow$', '$\rightarrow$', '$\forall$', '$\exists$' $\}$, the *predicate constants* $C = \{c_i^k \mid i,k \in \omega\}$, the *function variables*[1] $F = \{f_i^k \mid i,k \in \omega \wedge k > 0\}$, the *function constants*[2] $H = \{h_i^k \mid i,k \in \omega\}$, the *subject variables* $V = \{v_i \mid i \in \omega\}$, as well as *predicate variables* $P = \{p_i^k \mid i,k \in \omega\}$.[3] For the *arity* or *rank* of an operator we take the upper index. The set of predicate variables with zero arity is also called set of *proposition variables* or *sentence letters*: $A := \{p_i^0 \mid i \in \omega\}$. For subject variables we write short hand certain lower letters: $v_1 =$ 'u', $v_2 =$ 'v', $v_3 =$ 'w', $v_4 =$ 'x', $v_5 =$ 'y', $v_5 =$ 'z'. Furthermore we use the following short notations: for the predicate variables $p_1^n =$ 'ϕ' und $p_2^n =$ 'ψ', where the appropriate arity n is calculated by counting the subsequent parameters, for the proposition variables $a_1 =$ 'A', $a_2 =$ 'B' and $a_3 =$ 'C', for the function variables: $f_1^n =$ 'f' und $f_2^n =$ 'g', where again the appropriate arity n is calculated by counting the subsequent parameters. All binary propositional operators are written in infix notation. Parentheses surrounding groups of operands and operators are necessary to indicate the intended order in which operations are to be performed. E. g. for the operator $\wedge$ with the parameters A and B we write $(A \wedge B)$.

In the absence of parentheses the usual precedence rules determine the order of operations. Especially outermost parentheses are omitted.

[1] Function variables are used for a shorter notation. For example writing an identity proposition $x = y \rightarrow f(x) = f(y)$. Also this introduction prepares for the syntax extension for functional classes.

[2] Function constants are also introduced for convenience and are used for direct defined class functions. For example to define building of the power class operator, the union and intersection operator and the successor function. All these function constants can be interpreted as abbreviations.

[3] By ω we understand the natural numbers including zero. All involved symbols are pairwise disjoint. Therefore we can conclude for example: $f_i^k = f_{i'}^{k'} \rightarrow (k = k' \wedge i = i')$ and $h_i^k \neq v_j$.

The operators have the order of precedence described below (starting with the highest).

$$\begin{gathered}\neg, \forall, \exists \\ \wedge \\ \vee \\ \rightarrow, \leftrightarrow\end{gathered}$$

The term *term* is defined recursively as follows:

1. Every subject variable is a term.
2. Let $i, k \in \omega$ and let $t_1, \ldots, t_k$ be terms. Then $h_i^k(t_1, \ldots, t_k)$ is a term and if $k > 0$, so $f_i^k(t_1, \ldots, t_k)$ is a term too.

Therefore all zero arity function constants $\{h_i^0 \mid i \in \omega\}$ are terms. They are called *individual constants*.[4]

We define a *formula* and the relations *free* and *bound* subject variable recursivly as follows:

1. Every proposition variable is a formula. Such formulas contain no free or bound subject variables.
2. If p^k is a predicate variable with arity k and c^k is a predicate constant with arity k and $t_1, t_2, \ldots, t_k$ are terms, then $p^k(t_1, t_2, \ldots t_k)$ and $c^k(t_1, t_2, \ldots, t_k)$ are formulas. All subject variables that occur at least in one of $t_1, t_2, \ldots, t_k$ are free subject variables. Bound subject variables doesn't occur.[5]
3. Let α, β be formulas in which no subject variables occur bound in one formula and free in the other. Then $\neg\alpha$, $(\alpha \wedge \beta)$, $(\alpha \vee \beta)$, $(\alpha \rightarrow \beta)$ and $(\alpha \leftrightarrow \beta)$ are also formulas. Subject variables which occur free (respectively bound) in α or β stay free (respectively bound).
4. If in the formula α the subject variable x_1 occurs not bound[6], then also $\forall x_1\ \alpha$ and $\exists x_1\ \alpha$ are formulas. The symbol $\forall$ is called *universal quantifier* and $\exists$ as *existential quantifier*.

   Except for x_1 all free subject variables of α stay free. All bound subject variables are still bound and additionally x_1 is bound too.

All formulas that are only built by usage of 1. and 3. are called formulas of the *propositional calculus*.

For each formula α the following proposition holds: the set of free subject variables is disjoint with the set of bound subject variables..[7]

If a formula has the form $\forall x_1\ \alpha$ respectively $\exists x_1\ \alpha$ then the formula α is called the *scope* of the quantifier $\forall$ respectively $\exists$.

All formulas that are used to build up a formula by 1. to 4. are called *part formulas*.

[4] In an analogous manner subject variables might be defined as function variables of zero arity. Because subject variables play an importent role they have their own notation.

[5] This second item includes the first one, which is only listed for clarity.

[6] This means that x_1 is free in the formula or doesn't occur at all.

[7] Other formalizations allow for example $\forall x_1\ \alpha$ also if x_1 occurs already bound within α. Also propositions like $\alpha(x) \wedge (\forall x_1 \beta)$ are allowed. In this formalizations free and bound are defined for a single occurrence of a variable.

Chapter 2

Axioms and Rules of Inference

We now state the system of axioms for the predicate calculus and present the rules for obtaining new formulas from them.

2.1 Axioms

The logical operators of propositional calculus '$\neg$', '$\vee$', '$\wedge$', '$\leftrightarrow$' und '$\rightarrow$' combine arbitrary *propositions* to new propositions. A proposition is a statement that affirms or denies something and is either „true“ or „false“ (but not both).[1]

The new ingredient of predicate calculus is quantification.

The binary operator '$\vee$' (logical disjunction) combines the two propositions α and β into the new proposition $\alpha \vee \beta$. It results in true if at least one of its operands is true.

The unary operator '$\neg$' (logical negation) changes the truth value of a proposition α. $\neg\alpha$ has a value of true when its operand is false and a value of false when its operand is true.

The *logical implication* (*if*) the, *logical conjunction* (*and*) and the *logical equivalence* (*biconditional*) are defined as abbreviations. [2]

The logical implication ('if') could be defined as follows.

Definition 2.1 (Implication)**.**

$$\alpha \rightarrow \beta \ :\leftrightarrow \ \neg\alpha \ \vee \ \beta$$

The logical conjunction ('and') could be defined with de Morgan.

Definition 2.2 (Conjunction)**.**

$$\alpha \wedge \beta \ :\leftrightarrow \ \neg(\neg\alpha \ \vee \ \neg\beta)$$

The logical equivalence ('iff') is defined as usual.

Definition 2.3 (Equivalence)**.**

$$\alpha \wedge \beta \ :\leftrightarrow \ (\alpha \ \rightarrow \ \beta) \ \wedge \ (\beta \ \rightarrow \ \alpha)$$

[1] Later on we will define the symbols $\top$ and $\bot$ as truth values.

[2] Actually the symbols $\wedge, \rightarrow, \leftrightarrow$ are defined later on and are a syntax extension. But for convenience these symbols are already part of the logical language.

Now we come to the first axiom of propositional calculus. This axiom enables us to get rid of an unnecessary disjunction.

Axiom 1 (Disjunction Idempotence)**.**

$$(A \vee A) \rightarrow A$$

If a proposition is true, any alternative may be added without making it false.

Axiom 2 (Axiom of Weakening)**.**

$$A \rightarrow (A \vee B)$$

The disjunction should be commutative.

Axiom 3 (Commutativity of the Disjunction)**.**

$$(A \vee B) \rightarrow (B \vee A)$$

An disjunction could be added at both side of an implication.

Axiom 4 (Disjunctive Addition)**.**

$$(A \rightarrow B) \rightarrow ((C \vee A) \rightarrow (C \vee B))$$

If something is true for all x, it is true for any specific y.

Axiom 5 (Universal Instantiation)**.**

$$\forall x\ \phi(x) \rightarrow \phi(y)$$

If a predicate holds for some particular y, then there is an x for which the predicate holds.

Axiom 6 (Existential Generalization)**.**

$$\phi(y) \rightarrow \exists x\ \phi(x)$$

2.2 Rules of Inference

The following rules of inference enable us to obtain new true formulas from the axioms that are assumed to be true. From these new formulas we derive further formulas. So we can successively extend the set of true formulas.

Rule 1 (Modus Ponens)**.** *If both formulas α and $\alpha \rightarrow \beta$ are true, then we can conclude that β is true as well.*

Rule 2 (Replace Free Subject Variable)**.** *We start with a true formula. A free subject variable may be replaced by an arbitrary term, provided that the substituted term contains no subject variable that have a bound occurrence in the original formula. All occurrences of the free variable must be simultaneously replaced.*

The prohibition to use subject variables within the term that occur bound in the original formula has two reasons. First it ensures that the resulting formula is well-formed. Secondly it preserves the validity of the formula. Let us look at the following derivation.

$\forall x\ \exists y\ \phi(x,y)$	$\rightarrow$	$\exists y\ \phi(z,y)$	with axiom 5
$\forall x\ \exists y\ \phi(x,y)$	$\rightarrow$	$\exists y\ \phi(y,y)$	forbidden replacement: z in y, despite y is already bound
$\forall x\ \exists y\ x \neq y$	$\rightarrow$	$\exists y\ \neq y$	replace $\neq$ for ϕ

This last proposition is not valid in many models.

Rule 3 (Rename Bound Subject Variable)**.** *We may replace a bound subject variable occurring in a formula by any other subject variable, provided that the new variable occurs not free in the original formula. If the variable to be replaced occurs in more than one scope, then the replacement need be made in one scope only.*

Rule 4 (Replace Predicate Variable)**.** *Let α be a true formula that contains a predicate variable p of arity n, let $x_1, \ldots, x_n$ be subject variables and let $\beta(x_1, \ldots, x_n)$ be a formula where $x_1, \ldots, x_n$ are not bound. The formula $\beta(x_1, \ldots, x_n)$ must not contain all $x_1, \ldots, x_n$ as free subject variables. Furthermore it can also have other subject variables either free or bound.*

If the following conditions are fulfilled, then a replacement of all occurrences of $p(t_1, \ldots, t_n)$ each with appropriate terms $t_1, \ldots, t_n$ in α by $\beta(t_1, \ldots, t_n)$ results in another true formula.

- *the free variables of $\beta(x_1, \ldots, x_n)$ without $x_1, \ldots, x_n$ do not occur as bound variables in α*
- *each occurrence of $p(t_1, \ldots, t_n)$ in α contains no bound variable of $\beta(x_1, \ldots, x_n)$*
- *the result of the substitution is a well-formed formula*

See III §5 in [3].

The prohibition to use additional subject variables within the replacement formula that occur bound in the original formula assurs that the resulting formula is well-formed. Furthermore it preserves the validity of the formla. Take a look at the following derivation.

$\phi(x)$	$\rightarrow$	$\exists y\ \phi(y)$	with axiom 6
$(\exists y\ y = y) \wedge \phi(x)$	$\rightarrow$	$\exists y\ \phi(y)$	
$\exists y\ (y = y \wedge \phi(x))$	$\rightarrow$	$\exists y\ \phi(y)$	
$\exists y\ (y = y \wedge x \neq y)$	$\rightarrow$	$\exists y\ y \neq y$	forbidden replacment: $\phi(x)$ by $x \neq y$, despite y is already bound
$\exists y\ x \neq y$	$\rightarrow$	$\exists y\ y \neq y$	

The last proposition is not valid in many models.

Analogous to rule 4 we can replace function variables too.

Rule 5 (Replace Function Variable)**.** *Let α be an already proven formula that contains a function variable σ of arity n, let $x_1, \ldots, x_n$ be subject variables and let $\tau(x_1, \ldots, x_n)$ be an arbitrary term where $x_1, \ldots, x_n$ are not bound. The term $\tau(x_1, \ldots, x_n)$ must not contain all $x_1, \ldots, x_n$. as free subject variables. Furhermore it can also have other subject variables either free or bound.*

If the following conditions are fulfilled we can obtain a new true formula by replacing each occurrence of $\sigma(t_1, \ldots, t_n)$ with appropriate terms $t_1, \ldots, t_n$ in α by $\tau(t_1, \ldots, t_n)$.

- *the free variables of* $\tau(x_1, \ldots, x_n)$ *without* x_1, ..., x_n *do not occur as bound variables in* α
- *each occurrence of* $\sigma(x_1, \ldots, x_n)$ *in* α *contains no bound variable of* $\tau(x_1, \ldots, x_n)$
- *the result of the substitution is a well-formed formula*

Rule 6 (Universal Quantifier Introduction)**.** *If* $\alpha \rightarrow \beta(x_1)$ *is a true formula and* α *doesn't contain the subject variable* x_1*, then* $\alpha \rightarrow (\forall x_1\ (\beta(x_1)))$ *is a true formula too.*

Rule 7 (Existential Quantifier Introduction)**.** *If* $\alpha(x_1) \rightarrow \beta$ *is already proved to be true and* β *doesn't contain the subjekt variable* x_1*, then* $(\exists x_1\ \alpha(x_1)) \rightarrow \beta$ *is also a true formula.*

The usage and elimination of abbreviations and constants is also an inference rule. In many texts about mathematical logic these rules are not explicitly stated and this text is no exception. But in the exact QEDEQ format corresponding rules exist.

Chapter 3

Derived Propositions

Now we derive elementary propositions with the axioms and rules of inference of chapter 2.

3.1 Propositional Calculus

MISSING! OTHER: Zunächst behandeln wir die Aussagenlogik.

Definition 3.1. Die Prädikatskonstanten $\top$ für *true* oder *wahr* und $\bot$ für *false* oder *falsch* werden wie folgt definiert:

$$\top \quad \Leftrightarrow \quad A \vee \neg A \tag{3.1}$$
$$\bot \quad \Leftrightarrow \quad \neg \top \tag{3.2}$$

Die folgenden Formeln lassen sich beweisen.

Theorem 3.2.

$$\top \tag{3.3}$$
$$\neg\bot \tag{3.4}$$
$$A \rightarrow A \tag{3.5}$$
$$A \leftrightarrow A \tag{3.6}$$
$$A \vee B \leftrightarrow B \vee A \tag{3.7}$$
$$A \wedge B \leftrightarrow B \wedge A \tag{3.8}$$
$$A \wedge B \rightarrow A \tag{3.9}$$
$$(A \leftrightarrow B) \leftrightarrow (B \leftrightarrow A) \tag{3.10}$$
$$A \vee (B \vee C) \leftrightarrow (A \vee B) \vee C \tag{3.11}$$
$$A \wedge (B \wedge C) \leftrightarrow (A \wedge B) \wedge C \tag{3.12}$$
$$A \leftrightarrow A \vee A \tag{3.13}$$
$$A \leftrightarrow A \wedge A \tag{3.14}$$
$$A \leftrightarrow \neg\neg A \tag{3.15}$$
$$(A \rightarrow B) \leftrightarrow (\neg B \rightarrow \neg A) \tag{3.16}$$
$$(A \leftrightarrow B) \leftrightarrow (\neg A \leftrightarrow \neg B) \tag{3.17}$$
$$(A \rightarrow (B \rightarrow C)) \leftrightarrow (B \rightarrow (A \rightarrow C)) \tag{3.18}$$
$$\neg(A \vee B) \leftrightarrow \neg A \wedge \neg B \tag{3.19}$$
$$\neg(A \wedge B) \leftrightarrow \neg A \vee \neg B \tag{3.20}$$
$$A \vee (B \wedge C) \leftrightarrow (A \vee B) \wedge (A \vee C) \tag{3.21}$$
$$A \wedge (B \vee C) \leftrightarrow (A \wedge B) \vee (A \wedge C) \tag{3.22}$$
$$A \wedge \top \leftrightarrow A \tag{3.23}$$
$$A \wedge \bot \leftrightarrow \bot \tag{3.24}$$
$$A \vee \top \leftrightarrow \top \tag{3.25}$$
$$A \vee \bot \leftrightarrow A \tag{3.26}$$
$$A \vee \neg A \leftrightarrow \top \tag{3.27}$$
$$A \wedge \neg A \leftrightarrow \bot \tag{3.28}$$
$$(\top \rightarrow A) \leftrightarrow A \tag{3.29}$$
$$(\bot \rightarrow A) \leftrightarrow \top \tag{3.30}$$
$$(A \rightarrow \bot) \leftrightarrow \neg A \tag{3.31}$$
$$(A \rightarrow \top) \leftrightarrow \top \tag{3.32}$$
$$(A \leftrightarrow \top) \leftrightarrow A \tag{3.33}$$
$$(A \rightarrow B) \wedge (B \rightarrow C) \rightarrow A \rightarrow C \tag{3.34}$$
$$(A \leftrightarrow B) \wedge (B \leftrightarrow C) \rightarrow A \leftrightarrow C \tag{3.35}$$
$$((A \wedge B) \leftrightarrow (A \wedge C)) \leftrightarrow (A \rightarrow (B \leftrightarrow C)) \tag{3.36}$$
$$((A \wedge B) \leftrightarrow (A \wedge \neg B)) \leftrightarrow \neg A \tag{3.37}$$
$$(A \leftrightarrow (A \wedge B)) \leftrightarrow (A \leftrightarrow B) \tag{3.38}$$

3.2 Predicate Calculus

MISSING! OTHER: Für die Prädikatenlogik ergeben sich die folgenden Sätze.

Theorem 3.3.

$$\forall x\ (\phi(x) \to \psi(x)) \quad \to \quad \forall x\ (\phi(x)) \to \forall x\ (\psi(x)) \tag{3.39}$$
$$\forall x\ (\phi(x) \to \psi(x)) \quad \to \quad \exists x\ (\phi(x)) \to \exists x\ (\psi(x)) \tag{3.40}$$
$$\exists x\ (\phi(x) \wedge \psi(x)) \quad \to \quad \exists x\ (\phi(x)) \wedge \exists x\ (\psi(x)) \tag{3.41}$$
$$\forall x\ (\psi(x)) \vee \forall x\ (\psi(x)) \quad \to \quad \forall x\ (\phi(x) \vee \psi(x)) \tag{3.42}$$
$$\exists x\ (\phi(x) \vee \psi(x))) \quad \leftrightarrow \quad \exists x\ (\phi(x)) \vee \exists x\ (\psi(x)) \tag{3.43}$$
$$\forall x\ (\phi(x) \wedge \psi(x))) \quad \leftrightarrow \quad \forall x\ (\phi(x)) \wedge \forall x\ (\psi(x)) \tag{3.44}$$
$$\forall x\ \forall y\ (\phi(x,y)) \quad \leftrightarrow \quad \forall y\ \forall x\ (\phi(x,y)) \tag{3.45}$$
$$\exists x\ \exists y\ (\phi(x,y)) \quad \leftrightarrow \quad \exists y\ \exists x\ (\phi(x,y)) \tag{3.46}$$
$$\forall x\ (\phi(x) \to A) \quad \leftrightarrow \quad (\forall x\ (\phi(x)) \to A) \tag{3.47}$$
$$\forall x\ (A \to \phi(x)) \quad \leftrightarrow \quad (A \to \forall x\ (\phi(x))) \tag{3.48}$$
$$\forall x\ (\phi(x) \wedge A) \quad \leftrightarrow \quad \forall x\ (\phi(x)) \wedge A \tag{3.49}$$
$$\forall x\ (\phi(x) \vee A) \quad \leftrightarrow \quad (\forall x\ (\phi(x)) \vee A) \tag{3.50}$$
$$\forall x\ (\phi(x) \leftrightarrow A) \quad \leftrightarrow \quad (\forall x\ (\phi(x)) \leftrightarrow A) \tag{3.51}$$

+++ ergänzen

3.3 Derived Rules

MISSING! OTHER: Aus den logischen Grundlagen lassen sich logische Sätze und Metaregeln ableiten, die eine bequemere Argumentation ermöglichen. Erst mit diesem Regelwerk und zusätzlichen Definitionen und Abkürzungen wird die restliche Mathematik entwickelt. Dabei wird stets nur eine *konservative* Erweiterung der bisherigen Syntax vorgenommen. D. h. in dem erweiterten System lassen sich keine Formeln ableiten, die in der alten Syntax formuliert, aber dort nicht ableitbar sind. Im Folgenden werden solche Erweiterungen vorgestellt.

Rule 8 (Ersetzung durch logisch äquivalente Formeln)**.** *Sei die Aussage $\alpha \leftrightarrow \beta$ bereits bewiesen. Wird dann aus der Formel δ eine neue Formel γ dadurch gewonnen, dass ein beliebiges Vorkommen von α durch β ersetzt*[1] *wird und besitzt γ zumindest die freien Variablen (+++ unklar!) von δ, dann gilt $\delta \leftrightarrow \gamma$.*

Rule 9 (Allgemeine Assoziativität)**.** *Falls ein zweistelliger Operator das Assoziativitätsgesetz erfüllt, so erfüllt er auch das allgemeine Assoziativitätsgesetz. Dem Operator kann dann eine beliebige Stellenanzahl größer eins zugeschrieben werden. So wird beispielsweise anstelle für $(a+b)+(c+d)$ einfach $a+b+c+d$ geschrieben.*[2]

Rule 10 (Allgemeine Kommutativität)**.** *Falls ein Operator das allgemeine Assoziativitätsgesetz erfüllt und kommutativ ist, so sind alle Permutationen von Parameterreihenfolgen einander gleich oder äquivalent.*[3] *So gilt beispielsweise $a+b+c+d = c+a+d+b$.*

Definition 3.4 (Ableitbarkeit)**.** Eine Formel β heißt *aus der Formel α ableitbar*, wenn sich β mit Hilfe aller Regeln des Prädikatenkalküls und der um α vermehrten Gesamtheit aller wahren Formeln des Prädikatenkalküls herleitbar und $\alpha \to \beta$ eine Formel ist. Dabei dürfen die beiden Quantifizierungsregeln, die Einsetzungsregel für Prädikatenvariable und die Umbenennungsregel für freie

[1] Bei dieser Ersetzung kann es erforderlich sein, dass gebundene Variablen von β umbenannt werden müssen, damit sich wieder eine Formel ergibt.

[2] Der n-stellig Operator wird mit einer bestimmten Klammerung definiert, jede andere Klammerreihenfolge liefert jedoch dasselbe Ergebnis.

[3] Je nachdem ob es sich um einen Termoperator oder einen Formeloperator handelt.

Subjektvariable nur auf solche Variablen angewendet werden, die in der Formel α nicht auftreten.

Schreibweise: $\alpha \vdash \beta$.

Die Ableitbarkeit einer Formel β aus der Formel α ist streng zu unterscheiden von der Ableitbarkeit einer wahren Formel aus den Axiomen des Kalküls, denn im zweiten Fall stehen mehr Ableitungsregeln zur Verfügung. Falls beispielsweise die Formel A als Axiom aufgenommen wird, so ist die Formel $A \to B$ herleitbar. Hingegen läßt sich aus A nicht B ableiten.

Rule 11 (Deduktionstheorem)**.** *Wenn eine Formel β aus einer Formel α ableitbar ist, so ist die Formel $\alpha \to \beta$ im Prädikatenkalkül herleitbar.*

Chapter 4

Identy

MISSING! OTHER: +++ Fehlt noch

4.1 Identy Axioms

MISSING! OTHER: Es wird eine zweistellige Funktionskonstante festgelegt, welche in der Interpretation die Identität von Subjekten ausdrücken soll.

Definition 4.1 (Gleichheit)**.**

$$x = y \;\Leftrightarrow\; c_1^2(x, y)$$

Dazu werden zwei weitere Axiome, auch *Gleichheitsaxiome* genannt, formuliert.

Axiom 7 (Reflexivität der Gleichheit)**.**

$$x = x$$

Axiom 8 (Leibnizsche Ersetzbarkeit)**.**

$$x = y \rightarrow (\phi(x) \rightarrow \phi(y))$$

Theorem 4.2 (Symmetrie der Gleichheit)**.**

$$x = y \leftrightarrow y = x \tag{4.1}$$

Theorem 4.3 (Transitivität der Gleichheit)**.**

$$x = y \;\wedge\; y = z \rightarrow x = z \tag{4.2}$$

Theorem 4.4.

$$x = y \rightarrow (\phi(x) \leftrightarrow \phi(y)) \tag{4.3}$$

Theorem 4.5.

$$x = y \rightarrow f(x) = f(y) \tag{4.4}$$

4.2 ++ TODO Quantifiers

MISSING! OTHER: Bei der folgenden Definition muss die für $\psi(x)$ eingesetzte Formel „erkennen lassen“, über welche Subjektvariable quantifiziert wird. Das ist in der Regel darüber zu entscheiden, welche freie Subjektvariable als erstes in der Formel vorkommt.[1] In der exakten Syntax des Qedeq-Formats[2] ist die Subjektvariable immer angegeben.

[1] Beispielsweise ist in der folgenden Formel erkennbar, dass die zweite Quantifikation über die Subjektvariable m läuft: $\forall\, n \in \mathbb{N}\; \forall\, m \in n\; m < n$.

[2] Siehe unter http://www.qedeq.org/0_01_05/projektbeschreibung.pdf.

Definition 4.6 (Eingeschränkter Allquantor)**.**

$$\forall\ \psi(x)\ (\phi(x)) \ \Leftrightarrow\ \forall\ x\ (\psi(x) \rightarrow \phi(x))$$

Dazu passt die folgende Definition für den eingeschränkten Existenzquantor.[3]

Definition 4.7 (Eingeschränkter Existenzquantor)**.**

$$\exists\ \psi(x)\ (\phi(x)) \ \Leftrightarrow\ \exists\ x\ (\psi(x) \wedge \phi(x))$$

Für die Existenz genau eines Individuums mit einer bestimmten Eigenschaft wird nun ein gesonderter Quantor eingeführt.

Definition 4.8 (Eingeschränkter Existenzquantor für genau ein Individuum)**.**

$$\exists!\ \psi(x)\ (\phi(x)) \ \Leftrightarrow\ \exists\ \psi(x)\ (\phi(x) \wedge \forall\ \psi(y)\ (\phi(y) \rightarrow x = y))$$

Durch die Gültigkeit von $\exists!\ \psi(x)(\phi(x))$ kann daher eine Subjektkonstante definiert werden, wenn $\phi(x)$ und $\psi(x)$ durch Ausdrücke ersetzt werden, die ausser x keine freien Variablen, keine Prädikatsvariablen und keine Funktionsvariablen mehr enthält.

Rule 12 (Termdefinition durch Formel)**.** *Falls die Formel $\exists!x\ \alpha(x)$ gilt, dann kann die Termsyntax durch $D(x, \alpha(x))$ erweitert werden. Die Formel alpha(x) möge die Variable y nicht enthalten und $\beta(y)$ sei eine Formel, welche die Variable x nicht enthält. Dann wird durch $\beta(D(x, \alpha(x)))$ eine Formel definiert durch $\beta(y) \wedge \exists!x\ (\alpha(x) \wedge x = y)$. Auch in der abkürzenden Schreibweise gilt die Subjektvariable x als gebunden, die Subjektvariable y ist mit den obigen Einschränkungen frei wählbar und wird in der Abkürzung nicht weiter beachtet. Veränderungen von α in eine andere Formel α', die eventuell erforderlich sind, damit keine Variablenkollisionen mit Variablen aus β entstehen, müssen jedoch auch in der Abkürzung durchgeführt werden. Alle Termbildungsregeln werden entsprechend erweitert. Der Ausdruck ist auch ersetzbar durch $\exists!y\ (\beta(y) \wedge \alpha(y)$ oder durch $\beta(y) \wedge \alpha(y)$.*

Für eingeschränkte Quantoren gelten analog zu tsprechende Formeln.
+++

[3] Passend, da $\neg\forall\ \psi(x)\ (\phi(x)) \leftrightarrow \exists\ x\ \neg(\psi(x) \rightarrow \phi(x)) \leftrightarrow \exists\ x\ (\psi(x) \wedge \neg\phi(x)) \leftrightarrow \exists\ \psi(x)\ (\neg\phi(x))$.

Bibliography

[1] *A.N. Whitehead, B. Russell*, Principia Mathematica, Cambridge University Press, London 1910

[2] *P. Bernays*, Axiomatische Untersuchung des Aussagen-Kalkuls der „Principia Mathematica“, Math. Zeitschr. 25 (1926), 305-320

[3] *D. Hilbert, W. Ackermann*, Grundzüge der theoretischen Logik, 2nd ed., Berlin: Springer, 1938. English version: Principles of Mathematical Logic, Chelsea, New York 1950, ed. by R. E. Luce. See also http://www.math.uwaterloo.ca/~snburris/htdocs/scav/hilbert/hilbert.html 15

[4] *P.S. Novikov*, Elements of Mathematical Logic, Edinburgh: Oliver and Boyd, 1964.

[5] *E. Mendeslson*, Introduction to Mathematical Logic, 3rd. ed., Belmont, CA: Wadsworth, 1987.

[6] *V. Günther*, Lecture „Mathematik und Logik“, given at the University of Hamburg, 1994/1995.

[7] *M. Meyling*, Hilbert II, Presentation of Formal Correct Mathematical Knowledge, Basic Concept, http://www.qedeq.org/current/doc/project/qedeq_basic_concept_en.pdf.

[8] qedeq_set_theory_v1

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