First theorems of Propositional Calculus

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Abstract

This module includes first proofs of propositional calculus theorems.

Specification

This document has the following specification:

Name:	proptheo1
Version:	1.00.00
Rule version:	1.00.00
Orgin:	http://www.qedeq.org/0_00_53/proptheo1_1.00.00_1.00.00.qedeq

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References

This document uses the results of the following documents:

Name:	propaxiom
Version:	1.00.00
Rule version:	1.00.00
Orgin:	propaxiom_1.00.00_1.00.00.qedeq
pdf:	propaxiom_1.00.00_1.00.00.pdf

Content

First we prove a well known tautology:

Theorem 0.1 (theo1).

$$\neg P \ \lor \ P$$

Proof.

1	$(P \rightarrow Q) \rightarrow ((A \lor P) \rightarrow (A \lor Q))$	add axiom axiom4
2	$((P \lor P) \rightarrow Q) \rightarrow ((A \lor (P \lor P)) \rightarrow (A \lor Q))$	replace P by $P \lor P$ in 1
3	$((P \lor P) \rightarrow P) \rightarrow ((A \lor (P \lor P)) \rightarrow (A \lor P))$	replace Q by P in 2
4	$((P \lor P) \rightarrow P) \rightarrow ((\neg P \lor (P \lor P)) \rightarrow (\neg P \lor P))$	replace A by $\neg P$ in 3
5	$(P \lor P) \rightarrow P$	add axiom axiom1
6	$(\neg P \lor (P \lor P)) \rightarrow (\neg P \lor P)$	MP with 5, 4
7	$(P \rightarrow (P \lor P)) \rightarrow (\neg P \lor P)$	reverse abbreviation impl in 6 at occurence 1
8	$P \rightarrow (P \lor Q)$	add axiom axiom2
9	$P \rightarrow (P \lor P)$	replace Q by P in 8
10	$\neg P \lor P$	MP with 9, 7

□

We just use our first sentence to get the second theorem:

Theorem 0.2 (theo2).

$$P \rightarrow P$$

Proof.

1	$\neg P \lor P$	add sentence theo1
2	$P \rightarrow P$	reverse abbreviation impl in 1 at occurence 1

□

And another use of the first theorem, to get the law of the excluded middle (tertium non datur):

Theorem 0.3 (theo3).

$$P \lor \neg P$$

Proof.

1	$\neg P \lor P$	add sentence theo1
2	$(P \lor Q) \rightarrow (Q \lor P)$	add axiom axiom3
3	$(\neg P \lor Q) \rightarrow (Q \lor \neg P)$	replace P by $\neg P$ in 2
4	$(\neg P \lor P) \rightarrow (P \lor \neg P)$	replace Q by P in 3
5	$P \lor \neg P$	MP with 1, 4

□

Also trivial is:

Theorem 0.4 (theo4).

$$P \rightarrow \neg\neg P$$

Proof.

1	$P \lor \neg P$	add sentence theo3
2	$\neg P \lor \neg\neg P$	replace P by $\neg P$ in 1

3	$P \to \neg\neg P$	reverse abbreviation impl in 2 at occurence 1

□

Three negations:

Theorem 0.5 (theo5).

$$P \vee \neg\neg\neg P$$

Proof.

1	$P \to \neg\neg P$	add sentence theo4
2	$(P \to Q) \to ((A \vee P) \to (A \vee Q))$	add axiom axiom4
3	$(P \to \neg\neg P) \to ((A \vee P) \to (A \vee \neg\neg P))$	replace Q by $\neg\neg P$ in 2
4	$(A \vee P) \to (A \vee \neg\neg P)$	MP with 1, 3
5	$(A \vee \neg P) \to (A \vee \neg\neg\neg P)$	replace P by $\neg P$ in 4
6	$(P \vee \neg P) \to (P \vee \neg\neg\neg P)$	replace A by P in 5
7	$P \vee \neg P$	add sentence theo3
8	$P \vee \neg\neg\neg P$	MP with 7, 6

□

Now we could prove the reverse of Proposition 4:

Theorem 0.6 (theo6).

$$\neg\neg P \to P$$

Proof.

1	$P \vee \neg\neg\neg P$	add sentence theo5
2	$(P \vee Q) \to (Q \vee P)$	add axiom axiom3
3	$(P \vee \neg\neg\neg P) \to (\neg\neg\neg P \vee P)$	replace Q by $\neg\neg\neg P$ in 2
4	$\neg\neg\neg P \vee P$	MP with 1, 3
5	$\neg\neg P \to P$	reverse abbreviation impl in 4 at occurence 1

□