Further Theorems of Propositional Calculus

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Abstract

This module includes proofs of popositional calculus theorems. The following theorems and proofs are adapted from D. Hilbert and W. Ackermann's 'Grundzuege der theoretischen Logik' (Berlin 1928, Springer)

Specification

This document has the following specification:

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References

This document uses the results of the following documents:

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Version:	1.00.00
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Content

Negation of a conjunction:

Theorem 0.1 (hilb18).

$$\neg(P \wedge Q) \rightarrow (\neg P \vee \neg Q)$$

Proof.

1	$\neg\neg P \rightarrow P$	add sentence hilb6
2	$\neg\neg Q \rightarrow Q$	replace P by Q in 1
3	$\neg\neg(\neg P \vee \neg Q) \rightarrow (\neg P \vee \neg Q)$	replace Q by $\neg P \vee \neg Q$ in 2
4	$\neg(P \wedge Q) \rightarrow (\neg P \vee \neg Q)$	reverse abbreviation and in 3 at occurence 1

□

The reverse of a negation of a conjunction:

Theorem 0.2 (hilb19).

$$(\neg P \vee \neg Q) \rightarrow \neg(P \wedge Q)$$

Proof.

1	$P \rightarrow \neg\neg P$	add sentence hilb5
2	$Q \rightarrow \neg\neg Q$	replace P by Q in 1
3	$(\neg P \vee \neg Q) \rightarrow \neg\neg(\neg P \vee \neg Q)$	replace Q by $\neg P \vee \neg Q$ in 2
4	$(\neg P \vee \neg Q) \rightarrow \neg(P \wedge Q)$	reverse abbreviation and in 3 at occurence 1

□

Negation of a disjunction:

Theorem 0.3 (hilb20).

$$\neg(P \vee Q) \rightarrow (\neg P \wedge \neg Q)$$

Proof.

1	$P \rightarrow P$	add sentence hilb2
2	$Q \rightarrow Q$	replace P by Q in 1
3	$\neg(P \vee Q) \rightarrow \neg(P \vee Q)$	replace Q by $\neg(P \vee Q)$ in 2
4	$\neg(P \vee Q) \rightarrow \neg(\neg\neg P \vee Q)$	elementary equivalence in 3 at 8 of hilb5 with hilb5
5	$\neg(P \vee Q) \rightarrow \neg(\neg\neg P \vee \neg\neg Q)$	elementary equivalence in 4 at 11 of hilb5 with hilb5
6	$\neg(P \vee Q) \rightarrow (\neg P \wedge \neg Q)$	reverse abbreviation and in 5 at occurence 1

□

Reverse of a negation of a disjunction:

Theorem 0.4 (hilb21).

$$(\neg P \wedge \neg Q) \rightarrow \neg(P \vee Q)$$

Proof.

1	$P \to P$	add sentence hilb2
2	$Q \to Q$	replace P by Q in 1
3	$\neg(P \lor Q) \to \neg(P \lor Q)$	replace Q by $\neg(P \lor Q)$ in 2
4	$\neg(\neg\neg P \lor Q) \to \neg(P \lor Q)$	elementary equivalence in 3 at 4 of hilb5 with hilb5
5	$\neg(\neg\neg P \lor \neg\neg Q) \to \neg(P \lor Q)$	elementary equivalence in 4 at 7 of hilb5 with hilb5
6	$(\neg P \land \neg Q) \to \neg(P \lor Q)$	reverse abbreviation and in 5 at occurence 1

□

The Conjunction is commutative:

Theorem 0.5 (hilb22).

$$(P \land Q) \to (Q \land P)$$

Proof.

1	$P \to P$	add sentence hilb2
2	$Q \to Q$	replace P by Q in 1
3	$(P \land Q) \to (P \land Q)$	replace Q by $P \land Q$ in 2
4	$(P \land Q) \to \neg(\neg P \lor \neg Q)$	use abbreviation and in 3 at occurence 2
5	$(P \land Q) \to \neg(\neg Q \lor \neg P)$	elementary equivalence in 4 at 1 of hilb9 with hilb9
6	$(P \land Q) \to (Q \land P)$	reverse abbreviation and in 5 at occurence 1

□

A technical lemma that is simular to the previous one:

Theorem 0.6 (hilb23).

$$(Q \land P) \to (P \land Q)$$

Proof.

1	$(P \land Q) \to (Q \land P)$	add sentence hilb22
2	$(P \land A) \to (A \land P)$	replace Q by A in 1
3	$(B \land A) \to (A \land B)$	replace P by B in 2
4	$(B \land P) \to (P \land B)$	replace A by P in 3
5	$(Q \land P) \to (P \land Q)$	replace B by Q in 4

□

Reduction of a conjunction:

Theorem 0.7 (hilb24).

$$(P \land Q) \to P$$

Proof.

1	$P \to (P \lor Q)$	add axiom axiom2

2	$P \rightarrow (P \vee A)$	replace Q by A in 1
3	$B \rightarrow (B \vee A)$	replace P by B in 2
4	$B \rightarrow (B \vee \neg Q)$	replace A by $\neg Q$ in 3
5	$\neg P \rightarrow (\neg P \vee \neg Q)$	replace B by $\neg P$ in 4
6	$(P \rightarrow Q) \rightarrow (\neg Q \rightarrow \neg P)$	add sentence hilb7
7	$(P \rightarrow A) \rightarrow (\neg A \rightarrow \neg P)$	replace Q by A in 6
8	$(B \rightarrow A) \rightarrow (\neg A \rightarrow \neg B)$	replace P by B in 7
9	$(B \rightarrow (\neg P \vee \neg Q)) \rightarrow (\neg(\neg P \vee \neg Q) \rightarrow \neg B)$	replace A by $\neg P \vee \neg Q$ in 8
10	$(\neg P \rightarrow (\neg P \vee \neg Q)) \rightarrow (\neg(\neg P \vee \neg Q) \rightarrow \neg\neg P)$	replace B by $\neg P$ in 9
11	$\neg(\neg P \vee \neg Q) \rightarrow \neg\neg P$	MP with 5, 10
12	$(P \wedge Q) \rightarrow \neg\neg P$	reverse abbreviation and in 11 at occurence 1
13	$(P \wedge Q) \rightarrow P$	elementary equivalence in 12 at 1 of hilb6 with hilb6

□

Another form of a reduction of a conjunction:

Theorem 0.8 (hilb25).

$$(P \wedge Q) \rightarrow Q$$

Proof.

1	$(P \wedge Q) \rightarrow P$	add sentence hilb24
2	$(P \wedge A) \rightarrow P$	replace Q by A in 1
3	$(B \wedge A) \rightarrow B$	replace P by B in 2
4	$(B \wedge P) \rightarrow B$	replace A by P in 3
5	$(Q \wedge P) \rightarrow Q$	replace B by Q in 4
6	$(P \wedge Q) \rightarrow Q$	elementary equivalence in 5 at 1 of hilb22 with hilb22

□

The conjunction is associative too (first implication):

Theorem 0.9 (hilb26).

$$(P \wedge (Q \wedge A)) \rightarrow ((P \wedge Q) \wedge A)$$

Proof.

1	$((P \vee Q) \vee A) \rightarrow (P \vee (Q \vee A))$	add sentence hilb15
2	$(P \rightarrow Q) \rightarrow (\neg Q \rightarrow \neg P)$	add sentence hilb7
3	$(P \rightarrow A) \rightarrow (\neg A \rightarrow \neg P)$	replace Q by A in 2
4	$(B \rightarrow A) \rightarrow (\neg A \rightarrow \neg B)$	replace P by B in 3
5	$(B \rightarrow (P \vee (Q \vee A))) \rightarrow (\neg(P \vee (Q \vee A)) \rightarrow \neg B)$	replace A by $P \vee (Q \vee A)$ in 4
6	$(((P \vee Q) \vee A) \rightarrow (P \vee (Q \vee A))) \rightarrow (\neg(P \vee (Q \vee A)) \rightarrow \neg((P \vee Q) \vee A))$	replace B by $(P \vee Q) \vee A$ in 5
7	$\neg(P \vee (Q \vee A)) \rightarrow \neg((P \vee Q) \vee A)$	MP with 1, 6
8	$\neg(P \vee \neg\neg(Q \vee A)) \rightarrow \neg((P \vee Q) \vee A)$	elementary equivalence in 7 at 5 of hilb5 with hilb5
9	$\neg(P \vee \neg\neg(Q \vee A)) \rightarrow \neg(\neg\neg(P \vee Q) \vee A)$	elementary equivalence in 8 at 12 of hilb5 with hilb5
10	$\neg(P \vee \neg\neg(Q \vee B)) \rightarrow \neg(\neg\neg(P \vee Q) \vee B)$	replace A by B in 9

11	$\neg(P \vee \neg\neg(C \vee B)) \rightarrow \neg(\neg\neg(P \vee C) \vee B)$	replace Q by C in 10
12	$\neg(D \vee \neg\neg(C \vee B)) \rightarrow \neg(\neg\neg(D \vee C) \vee B)$	replace P by D in 11
13	$\neg(D \vee \neg\neg(C \vee \neg A)) \rightarrow \neg(\neg\neg(D \vee C) \vee \neg A)$	replace B by $\neg A$ in 12
14	$\neg(D \vee \neg\neg(\neg Q \vee \neg A)) \rightarrow \neg(\neg\neg(D \vee \neg Q) \vee \neg A)$	replace C by $\neg Q$ in 13
15	$\neg(\neg P \vee \neg\neg(\neg Q \vee \neg A)) \rightarrow \neg(\neg\neg(\neg P \vee \neg Q) \vee \neg A)$	replace D by $\neg P$ in 14
16	$(P \wedge \neg(\neg Q \vee \neg A)) \rightarrow \neg(\neg\neg(\neg P \vee \neg Q) \vee \neg A)$	reverse abbreviation and in 15 at occurence 1
17	$(P \wedge (Q \wedge A)) \rightarrow \neg(\neg\neg(\neg P \vee \neg Q) \vee \neg A)$	reverse abbreviation and in 16 at occurence 1
18	$(P \wedge (Q \wedge A)) \rightarrow (\neg(\neg P \vee \neg Q) \wedge A)$	reverse abbreviation and in 17 at occurence 1
19	$(P \wedge (Q \wedge A)) \rightarrow ((P \wedge Q) \wedge A)$	reverse abbreviation and in 18 at occurence 1

□

The conjunction is associative (second implication):

Theorem 0.10 (hilb27).

$$((P \wedge Q) \wedge A) \rightarrow (P \wedge (Q \wedge A))$$

Proof.

1	$(P \vee (Q \vee A)) \rightarrow ((P \vee Q) \vee A)$	add sentence hilb14
2	$(P \rightarrow Q) \rightarrow (\neg Q \rightarrow \neg P)$	add sentence hilb7
3	$(P \rightarrow A) \rightarrow (\neg A \rightarrow \neg P)$	replace Q by A in 2
4	$(B \rightarrow A) \rightarrow (\neg A \rightarrow \neg B)$	replace P by B in 3
5	$(B \rightarrow ((P \vee Q) \vee A)) \rightarrow (\neg((P \vee Q) \vee A) \rightarrow \neg B)$	replace A by $(P \vee Q) \vee A$ in 4
6	$((P \vee (Q \vee A)) \rightarrow ((P \vee Q) \vee A)) \rightarrow (\neg((P \vee Q) \vee A) \rightarrow \neg(P \vee (Q \vee A)))$	replace B by $P \vee (Q \vee A)$ in 5
7	$\neg((P \vee Q) \vee A) \rightarrow \neg(P \vee (Q \vee A))$	MP with 1, 6
8	$\neg(\neg\neg(P \vee Q) \vee A) \rightarrow \neg(P \vee (Q \vee A))$	elementary equivalence in 7 at 4 of hilb5 with hilb5
9	$\neg(\neg\neg(P \vee Q) \vee A) \rightarrow \neg(P \vee \neg\neg(Q \vee A))$	elementary equivalence in 8 at 13 of hilb5 with hilb5
10	$\neg(\neg\neg(P \vee Q) \vee B) \rightarrow \neg(P \vee \neg\neg(Q \vee B))$	replace A by B in 9
11	$\neg(\neg\neg(P \vee C) \vee B) \rightarrow \neg(P \vee \neg\neg(C \vee B))$	replace Q by C in 10
12	$\neg(\neg\neg(D \vee C) \vee B) \rightarrow \neg(D \vee \neg\neg(C \vee B))$	replace P by D in 11
13	$\neg(\neg\neg(D \vee C) \vee \neg A) \rightarrow \neg(D \vee \neg\neg(C \vee \neg A))$	replace B by $\neg A$ in 12
14	$\neg(\neg\neg(D \vee \neg Q) \vee \neg A) \rightarrow \neg(D \vee \neg\neg(\neg Q \vee \neg A))$	replace C by $\neg Q$ in 13
15	$\neg(\neg\neg(\neg P \vee \neg Q) \vee \neg A) \rightarrow \neg(\neg P \vee \neg\neg(\neg Q \vee \neg A))$	replace D by $\neg P$ in 14
16	$(\neg(\neg P \vee \neg Q) \wedge A) \rightarrow \neg(\neg P \vee \neg\neg(\neg Q \vee \neg A))$	reverse abbreviation and in 15 at occurence 1
17	$((P \wedge Q) \wedge A) \rightarrow \neg(\neg P \vee \neg\neg(\neg Q \vee \neg A))$	reverse abbreviation and in 16 at occurence 1
18	$((P \wedge Q) \wedge A) \rightarrow (P \wedge \neg(\neg Q \vee \neg A))$	reverse abbreviation and in 17 at occurence 1
19	$((P \wedge Q) \wedge A) \rightarrow (P \wedge (Q \wedge A))$	reverse abbreviation and in 18 at occurence 1

□

Form for the conjunction rule:

Theorem 0.11 (hilb28).

$$P \rightarrow (Q \rightarrow (P \wedge Q))$$

Proof.

1	$P \vee \neg P$	add sentence hilb4
2	$(\neg P \vee \neg Q) \vee \neg(\neg P \vee \neg Q)$	replace P by $\neg P \vee \neg Q$ in 1
3	$((P \vee Q) \vee A) \rightarrow (P \vee (Q \vee A))$	add sentence hilb15
4	$((P \vee Q) \vee B) \rightarrow (P \vee (Q \vee B))$	replace A by B in 3
5	$((P \vee C) \vee B) \rightarrow (P \vee (C \vee B))$	replace Q by C in 4
6	$((D \vee C) \vee B) \rightarrow (D \vee (C \vee B))$	replace P by D in 5
7	$((D \vee C) \vee \neg(\neg P \vee \neg Q)) \rightarrow (D \vee (C \vee \neg(\neg P \vee \neg Q)))$	replace B by $\neg(\neg P \vee \neg Q)$ in 6
8	$((D \vee \neg Q) \vee \neg(\neg P \vee \neg Q)) \rightarrow (D \vee (\neg Q \vee \neg(\neg P \vee \neg Q)))$	replace C by $\neg Q$ in 7
9	$((\neg P \vee \neg Q) \vee \neg(\neg P \vee \neg Q)) \rightarrow (\neg P \vee (\neg Q \vee \neg(\neg P \vee \neg Q)))$	replace D by $\neg P$ in 8
10	$\neg P \vee (\neg Q \vee \neg(\neg P \vee \neg Q))$	MP with 2, 9
11	$P \rightarrow (\neg Q \vee \neg(\neg P \vee \neg Q))$	reverse abbreviation impl in 10 at occurence 1
12	$P \rightarrow (Q \rightarrow \neg(\neg P \vee \neg Q))$	reverse abbreviation impl in 11 at occurence 1
13	$P \rightarrow (Q \rightarrow (P \wedge Q))$	reverse abbreviation and in 12 at occurence 1

□

Preconditions could be put together in a conjunction (first direction):

Theorem 0.12 (hilb29).

$$(P \rightarrow (Q \rightarrow A)) \rightarrow ((P \wedge Q) \rightarrow A)$$

Proof.

1	$P \rightarrow P$	add sentence hilb2
2	$Q \rightarrow Q$	replace P by Q in 1
3	$(P \rightarrow (Q \rightarrow A)) \rightarrow (P \rightarrow (Q \rightarrow A))$	replace Q by $P \rightarrow (Q \rightarrow A)$ in 2
4	$(P \rightarrow (Q \rightarrow A)) \rightarrow (\neg P \vee (Q \rightarrow A))$	use abbreviation impl in 3 at occurence 4
5	$(P \rightarrow (Q \rightarrow A)) \rightarrow (\neg P \vee (\neg Q \vee A))$	use abbreviation impl in 4 at occurence 4
6	$(P \rightarrow (Q \rightarrow A)) \rightarrow ((\neg P \vee \neg Q) \vee A)$	elementary equivalence in 5 at 1 of hilb14 with hilb14
7	$(P \rightarrow (Q \rightarrow A)) \rightarrow (\neg\neg(\neg P \vee \neg Q) \vee A)$	elementary equivalence in 6 at 8 of hilb5 with hilb5
8	$(P \rightarrow (Q \rightarrow A)) \rightarrow (\neg(\neg P \vee \neg Q) \rightarrow A)$	reverse abbreviation impl in 7 at occurence 1
9	$(P \rightarrow (Q \rightarrow A)) \rightarrow ((P \wedge Q) \rightarrow A)$	reverse abbreviation and in 8 at occurence 1

□

Preconditions could be put together in a conjunction (second direction):

Theorem 0.13 (hilb30).

$$((P \land Q) \to A) \to (P \to (Q \to A))$$

Proof.

1	$P \to P$	add sentence hilb2
2	$Q \to Q$	replace P by Q in 1
3	$(P \to (Q \to A)) \to (P \to (Q \to A))$	replace Q by $P \to (Q \to A)$ in 2
4	$(\neg P \lor (Q \to A)) \to (P \to (Q \to A))$	use abbreviation impl in 3 at occurence 2
5	$(\neg P \lor (\neg Q \lor A)) \to (P \to (Q \to A))$	use abbreviation impl in 4 at occurence 2
6	$((\neg P \lor \neg Q) \lor A) \to (P \to (Q \to A))$	elementary equivalence in 5 at 1 of hilb14 with hilb14
7	$(\neg\neg(\neg P \lor \neg Q) \lor A) \to (P \to (Q \to A))$	elementary equivalence in 6 at 3 of hilb5 with hilb5
8	$(\neg(\neg P \lor \neg Q) \to A) \to (P \to (Q \to A))$	reverse abbreviation impl in 7 at occurence 1
9	$((P \land Q) \to A) \to (P \to (Q \to A))$	reverse abbreviation and in 8 at occurence 1

□

Absorbtion of a conjunction (first direction):

Theorem 0.14 (hilb31).

$$(P \land P) \to P$$

Proof.

1	$(P \land Q) \to P$	add sentence hilb24
2	$(P \land P) \to P$	replace Q by P in 1

□

Absorbtion of a conjunction (second direction):

Theorem 0.15 (hilb32).

$$P \to (P \land P)$$

Proof.

1	$(P \lor P) \to P$	add sentence hilb11
2	$(P \to Q) \to (\neg Q \to \neg P)$	add sentence hilb7
3	$(P \to A) \to (\neg A \to \neg P)$	replace Q by A in 2
4	$(B \to A) \to (\neg A \to \neg B)$	replace P by B in 3
5	$(B \to P) \to (\neg P \to \neg B)$	replace A by P in 4
6	$((P \lor P) \to P) \to (\neg P \to \neg(P \lor P))$	replace B by $P \lor P$ in 5
7	$\neg P \to \neg(P \lor P)$	MP with 1, 6
8	$\neg Q \to \neg(Q \lor Q)$	replace P by Q in 7
9	$\neg\neg P \to \neg(\neg P \lor \neg P)$	replace Q by $\neg P$ in 8
10	$P \to \neg(\neg P \lor \neg P)$	elementary equivalence in 9 at 1 of hilb6 with hilb6

11	$P \rightarrow (P \land P)$	reverse abbreviation and in 10 at occurence 1

□

Absorbtion of identical preconditions (first direction):

Theorem 0.16 (hilb33).

$$(P \rightarrow (P \rightarrow Q)) \rightarrow (P \rightarrow Q)$$

Proof.

1	$(P \rightarrow (Q \rightarrow A)) \rightarrow ((P \land Q) \rightarrow A)$	add sentence hilb29
2	$(P \rightarrow (Q \rightarrow B)) \rightarrow ((P \land Q) \rightarrow B)$	replace A by B in 1
3	$(P \rightarrow (C \rightarrow B)) \rightarrow ((P \land C) \rightarrow B)$	replace Q by C in 2
4	$(D \rightarrow (C \rightarrow B)) \rightarrow ((D \land C) \rightarrow B)$	replace P by D in 3
5	$(D \rightarrow (C \rightarrow Q)) \rightarrow ((D \land C) \rightarrow Q)$	replace B by Q in 4
6	$(D \rightarrow (P \rightarrow Q)) \rightarrow ((D \land P) \rightarrow Q)$	replace C by P in 5
7	$(P \rightarrow (P \rightarrow Q)) \rightarrow ((P \land P) \rightarrow Q)$	replace D by P in 6
8	$(P \rightarrow (P \rightarrow Q)) \rightarrow (P \rightarrow Q)$	elementary equivalence in 7 at 1 of hilb31 with hilb31

□

Absorbtion of identical preconditions (second direction):

Theorem 0.17 (hilb34).

$$(P \rightarrow Q) \rightarrow (P \rightarrow (P \rightarrow Q))$$

Proof.

1	$((P \land Q) \rightarrow A) \rightarrow (P \rightarrow (Q \rightarrow A))$	add sentence hilb30
2	$((P \land Q) \rightarrow B) \rightarrow (P \rightarrow (Q \rightarrow B))$	replace A by B in 1
3	$((P \land C) \rightarrow B) \rightarrow (P \rightarrow (C \rightarrow B))$	replace Q by C in 2
4	$((D \land C) \rightarrow B) \rightarrow (D \rightarrow (C \rightarrow B))$	replace P by D in 3
5	$((D \land C) \rightarrow Q) \rightarrow (D \rightarrow (C \rightarrow Q))$	replace B by Q in 4
6	$((D \land P) \rightarrow Q) \rightarrow (D \rightarrow (P \rightarrow Q))$	replace C by P in 5
7	$((P \land P) \rightarrow Q) \rightarrow (P \rightarrow (P \rightarrow Q))$	replace D by P in 6
8	$(P \rightarrow Q) \rightarrow (P \rightarrow (P \rightarrow Q))$	elementary equivalence in 7 at 1 of hilb31 with hilb31

□

1 Cross Reference

This module is used by the following modules:

Name:	prophilbert3
Version:	1.00.00
Rule version:	1.02.00
Orgin:	`prophilbert3_1.00.00_1.02.00.qedeq`
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