First theorems of Propositional Calculus

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Abstract

This module includes proofs of popositional calculus theorems. The following theorems and proofs are adapted from D. Hilbert and W. Ackermann's 'Grundzuege der theoretischen Logik' (Berlin 1928, Springer)

Specification

This document has the following specification:

Name:	prophilbert1
Version:	1.00.00
Rule version:	1.02.00
Orgin:	http://www.qedeq.org/0_00_53/prophilbert1_1.00.00_1.02.00.qedeq

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References

This document uses the results of the following documents:

Name:	propaxiom
Version:	1.00.00
Rule version:	1.00.00
Orgin:	propaxiom_1.00.00_1.00.00.qedeq
pdf:	propaxiom_1.00.00_1.00.00.pdf
Name:	subst
Version:	1.00.00
Rule version:	1.01.00
Orgin:	subst_1.00.00_1.01.00.qedeq
pdf:	subst_1.00.00_1.01.00.pdf

Content

Now we could declare the rule *Apply Axiom.*
parameters:

p	proof line number
⟨link⟩	axiom reference

If the proof line n has the form 'p' and an axiom *ref* matches the form '$(p \rightarrow q)$' (p, q be formulas), then the string 'q' could be added as a new proof line.
For example: from proof line '$(r \vee s)$' we derive '$(r \vee s)$' by applying axiom 3.

Rule Declaration 0.1 (rule9). *Apply Axiom*

Analogous to the preceding we declare the rule *Apply Sentence.*

Rule Declaration 0.2 (rule10). *Apply Sentence*

First we prove a simple implication, that follows directly from the fourth axiom:

Theorem 0.1 (hilb1).

$$(P \rightarrow Q) \rightarrow ((A \rightarrow P) \rightarrow (A \rightarrow Q))$$

Proof.

1	$(P \rightarrow Q) \rightarrow ((A \vee P) \rightarrow (A \vee Q))$	add axiom axiom4
2	$(P \rightarrow Q) \rightarrow ((\neg A \vee P) \rightarrow (\neg A \vee Q))$	replace A by $\neg A$ in 1
3	$(P \rightarrow Q) \rightarrow ((A \rightarrow P) \rightarrow (\neg A \vee Q))$	reverse abbreviation impl in 2 at occurence 1
4	$(P \rightarrow Q) \rightarrow ((A \rightarrow P) \rightarrow (A \rightarrow Q))$	reverse abbreviation impl in 3 at occurence 1

□

This proposition is the form for the Hypothetical Syllogism.

Now we could declare the rule *Hypothetical Syllogism.*
parameters:

p	proof line number
m	proof line number

If the proof line n has the form '$(p \rightarrow q)$'; and the proof line m has the form '$(q \rightarrow r)$' (p, q and r must be formulas), then the string '$(p \rightarrow s)$' could be added as a new proof line.

Rule Declaration 0.3 (rule11). *Hypothetical Syllogism*

References, needed for declaration:
hilb1
The self implication could be derived:

Theorem 0.2 (hilb2).

$$P \rightarrow P$$

Proof.

1	$P \rightarrow (P \vee Q)$	add axiom axiom2
2	$P \rightarrow (P \vee P)$	replace Q by P in 1
3	$(P \vee P) \rightarrow P$	add axiom axiom1
4	$P \rightarrow P$	HS with 2 and 3

□

One form of the classical **tertium non datur**

Theorem 0.3 (hilb3).

$$\neg P \;\vee\; P$$

Proof.

1	$P \;\rightarrow\; P$	add sentence hilb2
2	$\neg P \;\vee\; P$	use abbreviation impl in 1 at occurence 1

□

The standard form of the excluded middle:

Theorem 0.4 (hilb4).

$$P \;\vee\; \neg P$$

Proof.

1	$\neg P \;\vee\; P$	add sentence hilb3
2	$P \;\vee\; \neg P$	apply axiom3 in 1

□

Double negation is implicated:

Theorem 0.5 (hilb5).

$$P \;\rightarrow\; \neg\neg P$$

Proof.

1	$P \;\vee\; \neg P$	add sentence hilb4
2	$\neg P \;\vee\; \neg\neg P$	replace P by $\neg P$ in 1
3	$P \;\rightarrow\; \neg\neg P$	reverse abbreviation impl in 2 at occurence 1

□

The reverse is also true:

Theorem 0.6 (hilb6).

$$\neg\neg P \;\rightarrow\; P$$

Proof.

1	$P \;\rightarrow\; \neg\neg P$	add sentence hilb5
2	$\neg P \;\rightarrow\; \neg\neg\neg P$	replace P by $\neg P$ in 1
3	$(P \;\vee\; \neg P) \;\rightarrow\; (P \;\vee\; \neg\neg\neg P)$	apply axiom4 in 2
4	$P \;\vee\; \neg P$	add sentence hilb4
5	$P \;\vee\; \neg\neg\neg P$	MP with 4, 3
6	$\neg\neg\neg P \;\vee\; P$	apply axiom3 in 5
7	$\neg\neg P \;\rightarrow\; P$	reverse abbreviation impl in 6 at occurence 1

□

The correct reverse of an implication:

Theorem 0.7 (hilb7).

$$(P \rightarrow Q) \rightarrow (\neg Q \rightarrow \neg P)$$

Proof.

1	$P \rightarrow \neg\neg P$	add sentence hilb5
2	$Q \rightarrow \neg\neg Q$	replace P by Q in 1
3	$(\neg P \vee Q) \rightarrow (\neg P \vee \neg\neg Q)$	apply axiom4 in 2
4	$(P \vee Q) \rightarrow (Q \vee P)$	add axiom axiom3
5	$(\neg P \vee \neg\neg Q) \rightarrow (\neg\neg Q \vee \neg P)$	Substitute Variables in 4
6	$(\neg P \vee Q) \rightarrow (\neg\neg Q \vee \neg P)$	HS with 3 and 5
7	$(P \rightarrow Q) \rightarrow (\neg\neg Q \vee \neg P)$	reverse abbreviation impl in 6 at occurence 1
8	$(P \rightarrow Q) \rightarrow (\neg Q \rightarrow \neg P)$	reverse abbreviation impl in 7 at occurence 1

□

Rule Declaration 0.4 (rule12). *Correct reverse of an implication*

References, needed for declaration:
hilb7

Rule Declaration 0.5 (rule13). *Add a Conjunction on the Left*

References, needed for declaration:
axiom4

Rule Declaration 0.6 (rule14). *Add a Conjunction on the Right*

References, needed for declaration:
axiom3 , axiom4
Definition of an Implication 1. part:

Theorem 0.8 (defimpl1).

$$(P \rightarrow Q) \rightarrow (\neg P \vee Q)$$

Proof.

1	$P \rightarrow P$	add sentence hilb2
2	$(P \rightarrow Q) \rightarrow (P \rightarrow Q)$	Substitute Variables in 1
3	$(P \rightarrow Q) \rightarrow (\neg P \vee Q)$	use abbreviation impl in 2 at occurence 3

□

Definition of an Implication 2. part:

Theorem 0.9 (defimpl2).

$$(\neg P \vee Q) \rightarrow (P \rightarrow Q)$$

Proof.

1	$P \rightarrow P$	add sentence hilb2
2	$(P \rightarrow Q) \rightarrow (P \rightarrow Q)$	Substitute Variables in 1
3	$(\neg P \vee Q) \rightarrow (P \rightarrow Q)$	use abbreviation impl in 2 at occurence 2

□

Rule Declaration 0.7 (rule17). *Addition of an Implication on the Right*

References, needed for declaration:
defimpl1 , defimpl2

Rule Declaration 0.8 (rule18). *Addition of an Implication on the Left*

References, needed for declaration:
defimpl1 , defimpl2
Definition of a Conjunction 1. part:

Theorem 0.10 (defand1).

$$(P \wedge Q) \rightarrow \neg(\neg P \vee \neg Q)$$

Proof.

1	$P \rightarrow P$	add sentence hilb2
2	$(P \wedge Q) \rightarrow (P \wedge Q)$	Substitute Variables in 1
3	$(P \wedge Q) \rightarrow \neg(\neg P \vee \neg Q)$	use abbreviation and in 2 at occurence 2

□

Definition of a Conjunction 2. part:

Theorem 0.11 (defand2).

$$\neg(\neg P \vee \neg Q) \rightarrow (P \wedge Q)$$

Proof.

1	$P \rightarrow P$	add sentence hilb2
2	$(P \wedge Q) \rightarrow (P \wedge Q)$	Substitute Variables in 1
3	$\neg(\neg P \vee \neg Q) \rightarrow (P \wedge Q)$	use abbreviation and in 2 at occurence 1

□

Rule Declaration 0.9 (rule19). *Addition of a Conjunction on the Left*

References, needed for declaration:
defand1 , defand2

Rule Declaration 0.10 (rule20). *Addition of a Conjunction on the Right*

References, needed for declaration:
defand1 , defand2
Definition of an Equivalence 1. part:

Theorem 0.12 (defequi1).

$$(P \leftrightarrow Q) \rightarrow ((P \rightarrow Q) \wedge (Q \rightarrow P))$$

Proof.

1	$P \to P$	add sentence hilb2
2	$(P \leftrightarrow Q) \to (P \leftrightarrow Q)$	Substitute Variables in 1
3	$(P \leftrightarrow Q) \to ((P \to Q) \wedge (Q \to P))$	use abbreviation equi in 2 at occurence 2

□

Definition of an Equivalence 2. part:

Theorem 0.13 (defequi2).

$$((P \to Q) \wedge (Q \to P)) \to (P \leftrightarrow Q)$$

Proof.

1	$P \to P$	add sentence hilb2
2	$(P \leftrightarrow Q) \to (P \leftrightarrow Q)$	Substitute Variables in 1
3	$((P \to Q) \wedge (Q \to P)) \to (P \leftrightarrow Q)$	use abbreviation equi in 2 at occurence 1

□

Rule Declaration 0.11 (rule21). *Addition of an Equivalence on the Left*

References, needed for declaration:
defequi1 , defequi2

Rule Declaration 0.12 (rule22). *Addition of an Equivalence on the Right*

References, needed for declaration:
defequi1 , defequi2

Rule Declaration 0.13 (rule30). *Elementary equivalence of two formulas*

A simular formulation for the second axiom:

Theorem 0.14 (hilb8).

$$P \to (Q \vee P)$$

Proof.

1	$P \to (P \vee Q)$	add axiom axiom2
2	$(P \vee Q) \to (Q \vee P)$	add axiom axiom3
3	$P \to (Q \vee P)$	HS with 1 and 2

□

A technical lemma (equal to the third axiom):

Theorem 0.15 (hilb9).

$$(P \vee Q) \to (Q \vee P)$$

Proof.

1	$(P \lor Q) \to (Q \lor P)$	add axiom axiom3

□

And another technical lemma (simular to the third axiom):

Theorem 0.16 (hilb10).

$$(Q \lor P) \to (P \lor Q)$$

Proof.

1	$(P \lor Q) \to (Q \lor P)$	add axiom axiom3
2	$(Q \lor P) \to (P \lor Q)$	Substitute Variables in 1

□

A technical lemma (equal to the first axiom):

Theorem 0.17 (hilb11).

$$(P \lor P) \to P$$

Proof.

1	$(P \lor P) \to P$	add axiom axiom1

□

A simple propositon that follows directly from the second axiom:

Theorem 0.18 (hilb12).

$$P \to (P \lor P)$$

Proof.

1	$P \to (P \lor Q)$	add axiom axiom2
2	$P \to (P \lor P)$	replace Q by P in 1

□

This is a pre form for the associative law:

Theorem 0.19 (hilb13).

$$(P \lor (Q \lor A)) \to (Q \lor (P \lor A))$$

Proof.

1	$P \to (Q \lor P)$	add sentence hilb8
2	$A \to (P \lor A)$	Substitute Variables in 1
3	$(Q \lor A) \to (Q \lor (P \lor A))$	apply axiom4 in 2
4	$(P \lor (Q \lor A)) \to (P \lor (Q \lor (P \lor A)))$	apply axiom4 in 3
5	$(P \lor (Q \lor A)) \to ((Q \lor (P \lor A)) \lor P)$	elementary equivalence in 4 at 3 of hilb9 with hilb9
6	$(P \lor A) \to (Q \lor (P \lor A))$	replace P by $P \lor A$ in 1
7	$P \to (P \lor Q)$	add axiom axiom2
8	$P \to (P \lor A)$	replace Q by A in 7

9	$P \rightarrow (Q \vee (P \vee A))$	HS with 8 and 6
10	$((Q \vee (P \vee A)) \vee P) \rightarrow ((Q \vee (P \vee A)) \vee (Q \vee (P \vee A)))$	apply axiom4 in 9
11	$((Q \vee (P \vee A)) \vee P) \rightarrow (Q \vee (P \vee A))$	elementary equivalence in 10 at 1 of hilb11 with hilb11
12	$(P \vee (Q \vee A)) \rightarrow (Q \vee (P \vee A))$	HS with 5 and 11

□

The associative law for the disjunction (first direction):

Theorem 0.20 (hilb14).

$$(P \vee (Q \vee A)) \rightarrow ((P \vee Q) \vee A)$$

Proof.

1	$P \rightarrow P$	add sentence hilb2
2	$(P \vee (Q \vee A)) \rightarrow (P \vee (Q \vee A))$	Substitute Variables in 1
3	$(P \vee (Q \vee A)) \rightarrow (P \vee (A \vee Q))$	elementary equivalence in 2 at 4 of hilb9 with hilb9
4	$(P \vee (Q \vee A)) \rightarrow (Q \vee (P \vee A))$	add sentence hilb13
5	$(P \vee (A \vee Q)) \rightarrow (A \vee (P \vee Q))$	Substitute Variables in 4
6	$(P \vee (Q \vee A)) \rightarrow (A \vee (P \vee Q))$	HS with 3 and 5
7	$(P \vee (Q \vee A)) \rightarrow ((P \vee Q) \vee A)$	elementary equivalence in 6 at 3 of hilb9 with hilb9

□

The associative law for the disjunction (second direction):

Theorem 0.21 (hilb15).

$$((P \vee Q) \vee A) \rightarrow (P \vee (Q \vee A))$$

Proof.

1	$(P \vee (Q \vee A)) \rightarrow ((P \vee Q) \vee A)$	add sentence hilb14
2	$(A \vee (Q \vee P)) \rightarrow ((A \vee Q) \vee P)$	Substitute Variables in 1
3	$((Q \vee P) \vee A) \rightarrow ((A \vee Q) \vee P)$	elementary equivalence in 2 at 1 of hilb9 with hilb9
4	$((P \vee Q) \vee A) \rightarrow ((A \vee Q) \vee P)$	elementary equivalence in 3 at 2 of hilb9 with hilb9
5	$((P \vee Q) \vee A) \rightarrow (P \vee (A \vee Q))$	elementary equivalence in 4 at 3 of hilb9 with hilb9
6	$((P \vee Q) \vee A) \rightarrow (P \vee (Q \vee A))$	elementary equivalence in 5 at 4 of hilb9 with hilb9

□

Another consequence from hilb13 is the exchange of preconditions:

Theorem 0.22 (hilb16).

$$(P \rightarrow (Q \rightarrow A)) \rightarrow (Q \rightarrow (P \rightarrow A))$$

Proof.

1	$(P \lor (Q \lor A)) \to (Q \lor (P \lor A))$	add sentence hilb13
2	$(\neg P \lor (\neg Q \lor A)) \to (\neg Q \lor (\neg P \lor A))$	Substitute Variables in 1
3	$(P \to (\neg Q \lor A)) \to (\neg Q \lor (\neg P \lor A))$	reverse abbreviation impl in 2 at occurence 1
4	$(P \to (Q \to A)) \to (\neg Q \lor (\neg P \lor A))$	reverse abbreviation impl in 3 at occurence 1
5	$(P \to (Q \to A)) \to (Q \to (\neg P \lor A))$	reverse abbreviation impl in 4 at occurence 1
6	$(P \to (Q \to A)) \to (Q \to (P \to A))$	reverse abbreviation impl in 5 at occurence 1

□

An analogus form for hilb16:

Theorem 0.23 (hilb17).

$$(Q \to (P \to A)) \to (P \to (Q \to A))$$

Proof.

1	$(P \to (Q \to A)) \to (Q \to (P \to A))$	add sentence hilb16
2	$(Q \to (P \to A)) \to (P \to (Q \to A))$	Substitute Variables in 1

□

1 Cross Reference

This module is used by the following modules:

Name:	prophilbert2
Version:	1.00.00
Rule version:	1.02.00
Orgin:	`prophilbert2_1.00.00_1.02.00.qedeq`
pdf:	`prophilbert2_1.00.00_1.02.00.pdf`