Some more theorems of Predicate Calculus

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Abstract

This module includes first proofs of predicate calculus theorems.

Specification

This document has the following specification:

Name:	predtheo2
Version:	1.00.00
Rule version:	1.02.00
Orgin:	http://www.qedeq.org/0_00_53/predtheo2_1.00.00_1.02.00.qedeq

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References

This document uses the results of the following documents:

Name:	predaxiom
Version:	1.00.00
Rule version:	1.00.00
Orgin:	predaxiom_1.00.00_1.00.00.qedeq
pdf:	predaxiom_1.00.00_1.00.00.pdf
Name:	prophilbert3
Version:	1.00.00
Rule version:	1.02.00
Orgin:	prophilbert3_1.00.00_1.02.00.qedeq
pdf:	prophilbert3_1.00.00_1.02.00.pdf

Content

A simple implication:

Theorem 0.1 (predtheo1).

$$\forall x\, R(x) \;\rightarrow\; \exists x\, R(x)$$

Proof.

1	$\forall x\, R(x) \;\rightarrow\; R(y)$	add axiom axiom5
2	$R(y) \;\rightarrow\; \exists x\, R(x)$	add axiom axiom6
3	$\forall x\, R(x) \;\rightarrow\; \exists x\, R(x)$	HS with 1 and 2

□

A well known implication:

Theorem 0.2 (predtheo2).

$$\exists x\, R(x) \;\rightarrow\; \neg\forall x\, \neg R(x)$$

Proof.

1	$\forall x\, R(x) \;\rightarrow\; R(y)$	add axiom axiom5
2	$\forall x\, \neg R(x) \;\rightarrow\; \neg R(y)$	replace $R(@S_1)$ by $\neg R(@S_1)$ in 1
3	$\neg\forall x\, \neg R(x) \;\vee\; \neg R(y)$	use abbreviation impl in 2 at occurence 1
4	$(P \;\vee\; Q) \;\rightarrow\; (Q \;\vee\; P)$	add axiom axiom3
5	$(\neg\forall x\, \neg R(x) \;\vee\; Q) \;\rightarrow\; (Q \;\vee\; \neg\forall x\, \neg R(x))$	replace P by $\neg\forall x\, \neg R(x)$ in 4
6	$(\neg\forall x\, \neg R(x) \;\vee\; \neg R(y)) \;\rightarrow\; (\neg R(y) \;\vee\; \neg\forall x\, \neg R(x))$	replace Q by $\neg R(y)$ in 5
7	$\neg R(y) \;\vee\; \neg\forall x\, \neg R(x)$	MP with 3, 6
8	$R(y) \;\rightarrow\; \neg\forall x\, \neg R(x)$	reverse abbreviation impl in 7 at occurence 1
9	$\exists y\, R(y) \;\rightarrow\; \neg\forall x\, \neg R(x)$	Particularize by y in 8
10	$\exists x\, R(x) \;\rightarrow\; \neg\forall x\, \neg R(x)$	rename y into x in 9 at occurence 1

□

The reverse is also true:

Theorem 0.3 (predtheo3).

$$\neg\forall x\, \neg R(x) \;\rightarrow\; \exists x\, R(x)$$

Proof.

1	$R(y) \;\rightarrow\; \exists x\, R(x)$	add axiom axiom6
2	$\neg\exists x\, R(x) \;\rightarrow\; \neg R(y)$	apply hilb7 in 1
3	$\neg\exists x\, R(x) \;\rightarrow\; \forall y\, \neg R(y)$	Generalize by y in 2
4	$\neg\forall y\, \neg R(y) \;\rightarrow\; \neg\neg\exists x\, R(x)$	apply hilb7 in 3
5	$\neg\forall y\, \neg R(y) \;\rightarrow\; \exists x\, R(x)$	elementary equivalence in 4 at 1 of hilb6 with hilb6
6	$\neg\forall x\, \neg R(x) \;\rightarrow\; \exists x\, R(x)$	rename y into x in 5 at occurence 1

□

Exchange of universal quantors:

Theorem 0.4 (predtheo4).

$$\forall x\,\forall y\,R(x,y) \;\rightarrow\; \forall y\,\forall x\,R(x,y)$$

Proof.

1	$\forall x\,R(x) \;\rightarrow\; R(y)$	add axiom axiom5
2	$\forall y\,R(y) \;\rightarrow\; R(u)$	Substitute Variables in 1
3	$\forall y\,R(z,y) \;\rightarrow\; R(z,u)$	replace $R(@S_1)$ by $R(z,@S_1)$ in 2
4	$\forall v\,R(v) \;\rightarrow\; R(z)$	Substitute Variables in 1
5	$\forall v\,\forall w\,R(v,w) \;\rightarrow\; \forall w\,R(z,w)$	replace $R(@S_1)$ by $\forall w\;R(@S_1,w)$ in 4
6	$\forall x\,\forall y\,R(x,y) \;\rightarrow\; \forall y\,R(z,y)$	Substitute Variables in 5
7	$\forall x\,\forall y\,R(x,y) \;\rightarrow\; R(z,u)$	HS with 6 and 3
8	$\forall x\,\forall y\,R(x,y) \;\rightarrow\; \forall z\,R(z,u)$	Generalize by z in 7
9	$\forall x\,\forall y\,R(x,y) \;\rightarrow\; \forall u\,\forall z\,R(z,u)$	Generalize by u in 8
10	$\forall x\,\forall y\,R(x,y) \;\rightarrow\; \forall y\,\forall z\,R(z,y)$	rename u into y in 9 at occurence 1
11	$\forall x\,\forall y\,R(x,y) \;\rightarrow\; \forall y\,\forall x\,R(x,y)$	rename z into x in 10 at occurence 1

□

Implication of changing sequence of existence and universal quantor:

Theorem 0.5 (predtheo5).

$$\exists x\,\forall y\,R(x,y) \;\rightarrow\; \forall y\,\exists x\,R(x,y)$$

Proof.

1	$\forall x\,R(x) \;\rightarrow\; R(y)$	add axiom axiom5
2	$\forall y\,R(y) \;\rightarrow\; R(u)$	Substitute Variables in 1
3	$\forall y\,R(x,y) \;\rightarrow\; R(x,u)$	replace $R(@S_1)$ by $R(x,@S_1)$ in 2
4	$R(y) \;\rightarrow\; \exists x\,R(x)$	add axiom axiom6
5	$R(x) \;\rightarrow\; \exists z\,R(z)$	Substitute Variables in 4
6	$R(x,u) \;\rightarrow\; \exists z\,R(z,u)$	replace $R(@S_1)$ by $R(@S_1,u)$ in 5
7	$\forall y\,R(x,y) \;\rightarrow\; \exists z\,R(z,u)$	HS with 3 and 6
8	$\exists x\,\forall y\,R(x,y) \;\rightarrow\; \exists z\,R(z,u)$	Particularize by x in 7
9	$\exists x\,\forall y\,R(x,y) \;\rightarrow\; \forall u\,\exists z\,R(z,u)$	Generalize by u in 8
10	$\exists x\,\forall y\,R(x,y) \;\rightarrow\; \forall y\,\exists x\,R(x,y)$	Substitute Variables in 9

□